

Lattice Calculations of $B \rightarrow K/K^* l^+ l^-$ form factors

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1 Introduction

The exclusive rare semileptonic decays $B \rightarrow Kl^+l^-$, $B \rightarrow K^*l^+l^-$, and the corresponding quark level process $b \rightarrow sl^+l^-$ are mediated by flavor changing neutral currents (FCNC). In the standard model such processes arise at one-loop and are therefore suppressed. Hence, these decays are good candidates to search for new physics beyond the SM. Experimental measurements of $B \rightarrow K/K^*l^+l^-$ decays have been performed by the BABAR, Belle, and CDF Collaborations [1, 2, 3]. Most recently, new results have been reported by LHCb [4, 5, 6]. We expect to see increasingly accurate results from both LHCb and from planned high-intensity B factories [7].

To find evidence of new physics, it is necessary to compare the experimental results with theoretical predictions from the SM. The current experimental results are consistent with the SM predictions, but when the experimental error decreases, we need more accurate theoretical predictions. The accuracy of the theoretical prediction in $B \rightarrow K/K^*l^+l^-$ is limited by the error of the hadronic matrix elements $\langle B | \hat{O} | K/K^* \rangle$, which are parameterized by form factors. Some earlier theoretical calculations were based on the form factors calculated from Light Cone Sum Rules (LCSR). LCSR computes the form factors at low q^2 , where q^2 is the outgoing dilepton invariant mass squared. The form factors at large q^2 are extrapolated from low q^2 results, which ends in a large error. Lattice QCD can calculate form factors directly at large q^2 from first principles. Some earlier lattice QCD works calculated $B \rightarrow K/K^*l^+l^-$ form factors within quenched approximation [8, 9, 10, 11, 12, 13, 14]. These works obtained form factors at large q^2 within 15%-20% accuracy. Moreover, the quenched calculations have the systematic error from lack of sea quark, which is difficult to remove. Modern lattice QCD simulations include realistic sea quark effects, hence removing this systematic error. Recently, three lattice collaborations (FNAL/MILC, HPQCD, and Cambridge/W&M/Edinburgh) have started calculations of $B \rightarrow K/K^*l^+l^-$ form factors based on the 2+1 flavor ensembles generated by the MILC collaboration [15]. The FNAL/MILC collaborations and HPQCD collaboration are working on the $B \rightarrow Kl^+l^-$ decay [17, 16, 18]. The Cambridge/W&M/Edinburgh collaboration are working on both of these two decays [19, 20].

Current methods in lattice QCD allow for calculations of $B \rightarrow Kl^+l^-$ with complete control over all sources of systematic error. This is not the case for $B \rightarrow K^*l^+l^-$, however, due to the fact that the K^* meson is unstable, and needs to be treated as a resonance. Hence current lattice QCD calculations of this process contain an additional systematic error that is difficult to quantify [19].

2 Lattice formalism

The three new lattice QCD calculations use $N_f=2+1$ flavor gauge configurations generated by the MILC Collaboration [21]. The MILC collaboration used the tree-level improved Lüscher-Wise action for the gauge fields and asqtad-improved staggered action for light sea quarks. These improvements suppress the lattice artifacts to the order of $\mathcal{O}(\alpha_s a^2)$ for the gluon field and $\mathcal{O}(\alpha_s a^2)$, $\mathcal{O}(a^4)$ for the fermion field [21], where a denotes the lattice spacing. These ensembles have four lattice spacings which are about 0.12fm, 0.09fm, 0.06fm, and 0.045fm. The FNAL/MILC collaborations employ ensembles on these four lattice spacings. The HPQCD collaboration and Cambridge/W&M/Edinburgh collaboration employ the ensembles with lattice spacings at $a \approx 0.12\text{fm}$ and $a \approx 0.09\text{fm}$. Although these three groups use the similar gauge ensembles, they employ different actions for valence b and s quarks. Both FNAL/MILC and Cambridge/W&M/Edinburgh group use the asqtad-improved staggered action for the valence strange quark. HPQCD uses HISQ action for valence strange quark [22]. The HISQ action is more improved than the asqtad action [22, 21]. For the heavy quark, the FNAL/MILC collaborations use the Sheikholeslami-Wohlert (SW) action [23] with the Fermilab interpretation for the b quark [24]. This action can be systematically improved and FNAL/MILC collaborations tune the b quark hopping parameter κ and clover coefficient c_{SW} to remove the discretization errors through next-to-leading order ($\mathcal{O}(1/m_b)$) [25, 26]. The HPQCD collaboration and Cambridge/W&M/Edinburgh collaboration use (moving)-NRQCD method [27, 28] for the b quark. The NRQCD method expands the relativistic QCD action by the order of v_b which is the velocity of the b quark. The heavy quark action is tuned to include the $\mathcal{O}(\Lambda_{\text{QCD}}^2/m_b^2)$ corrections. Because these three groups are working on similar quantities with different discretization methods, it provides a good cross check for their results.

3 The lattice QCD calculation on form factors

The theoretical description of the $B \rightarrow K/K^*l^+l^-$ process is based on the Operator Production Expansion (OPE). The low energy effective Hamiltonian for the $b \rightarrow sl^+l^-$

transition is [30, 31, 32, 33]

$$\mathcal{H}_{eff} = -\frac{4G_F}{\sqrt{2}}V_{tb}V_{ts}^* \sum_i \mathcal{C}_i(\mu)\mathcal{O}_i(\mu) \quad (1)$$

where \mathcal{O}_i s are four-fermion operators of dimension six. \mathcal{C}_i s are the corresponding Wilson coefficients. Most of the SM contribution is from the operators $\mathcal{O}_{7,9,10}$. The operator \mathcal{O}_7 is a photon dipole operator and $\mathcal{O}_{9,10}$ are semileptonic operators. Theoretical predictions are calculated from \mathcal{H}_{eff} and contain hadron matrix elements that are parametrized by form factors. For $B \rightarrow Kl^+l^-$ decay, there are three form factors f_+ , f_0 , and f_T

$$\langle K | i\bar{s}\gamma^\mu b | B \rangle = f_+(q^2) \left(p_B^\mu + p_K^\mu - \frac{M_B^2 - M_K^2}{q^2} q^\mu \right) + f_0(q^2) \frac{M_B^2 - M_K^2}{q^2} q^\mu, \quad (2)$$

$$\langle K | i\bar{s}\sigma^{\mu\nu} b | B \rangle = \frac{2f_T(q^2)}{M_B + M_K} (p^\mu k^\nu - p^\nu k^\mu) q_\nu, \quad (3)$$

where p_B and p_K are the B meson and kaon momenta, respectively. FNAL/MILC collaborations and HPQCD calculate these two matrix elements in the B meson rest frame. The q^2 becomes $M_B^2 + M_K^2 - 2M_B E_K$ in this reference frame. The matrix elements are reparametrized as

$$\langle K | i\bar{s}\gamma^\mu b | B \rangle = \sqrt{2M_B} [v^\mu f_\parallel(E_K) + p_\perp^\mu f_\perp(E_K)], \quad (4)$$

where $v^\mu = p_B^\mu/M_B$ is the four-velocity of the B meson and $p_\perp^\mu = p_K^\mu - (p_K \cdot v)v^\mu$. The form factors f_\parallel and f_\perp are solved from the temporal and spatial components of the matrix element of the vector current. Finally, the form factors (f_+, f_0) are reconstructed from f_\parallel and f_\perp by

$$f_+ = \frac{1}{\sqrt{2M_B}} [f_\parallel + (M_B - E_K)f_\perp], \quad (5)$$

$$f_0 = \frac{\sqrt{2M_B}}{M_B^2 - M_K^2} [(M_B - E_K)f_\parallel + (E_K^2 - M_K^2)f_\perp]. \quad (6)$$

A similar method can be applied to f_T :

$$f_T = \frac{M_B + M_K}{\sqrt{2M_B}} \frac{\langle K | i b \sigma^{0i} s | B \rangle}{\sqrt{2M_B} p_K^i}. \quad (7)$$

For $B \rightarrow K^* l^+ l^-$ decay, there are four hadronic matrix elements related to the

theoretical predictions. They are

$$\langle K^*(k, \epsilon) | \bar{s} \gamma^\mu b | B(p) \rangle = \frac{2iV(q^2)}{M_B + M_{K^*}} \epsilon^{\mu\nu\rho\sigma} \epsilon_\nu^* k_\rho p'_\sigma \quad (8)$$

$$\begin{aligned} \langle K^*(k, \epsilon) | \bar{s} \gamma^\mu \gamma_5 b | B(p) \rangle &= 2M_{K^*} A_0(q^2) \frac{\epsilon^* \cdot q}{q^2} q^\mu + (M_B + M_{K^*}) A_1(q^2) (\epsilon^{*\mu} - \frac{\epsilon^* \cdot q}{q^2} q^\mu) \\ &\quad - A_2(q^2) \frac{\epsilon^* \cdot q}{M_B + M_{K^*}} (p^\mu + k^\mu - \frac{M_B^2 - M_{K^*}^2}{q^2} q^\mu) \end{aligned} \quad (9)$$

$$q^\nu \langle K^*(k, \epsilon) | \bar{s} \sigma_{\mu\nu} b | B(p) \rangle = 4T_1(q^2) \epsilon_{\mu\nu\kappa\rho} \epsilon_\rho^* p_\kappa k_\sigma \quad (10)$$

$$\begin{aligned} q^\nu \langle K^*(k, \epsilon) | \bar{s} \sigma_{\mu\nu} \gamma_5 b | B(p) \rangle &= 2iT_2(q^2) [\epsilon_\mu^* (M_B^2 - M_{K^*}^2) - (\epsilon^* \cdot q)(p + k)_\mu] \\ &\quad 2iT_3(q^2) (\epsilon^* \cdot q) [q_\mu - \frac{q^2}{M_B^2 - M_{K^*}^2} (p + k)_\mu] \end{aligned} \quad (11)$$

Like the $f_{+,0,T}$ case, lattice QCD calculates the form factors by calculating a particular component of the matrix elements. Details on $B \rightarrow K^* l^+ l^-$ form factors are in Ref. [19, 20].

4 Preliminary results on $B \rightarrow K/K^* l^+ l^-$ form factors

4.1 Extract form factors on the lattices

First of all, we extract the meson masses and energies from the two-point correlation functions measured on the gauge configurations:

$$C_2(t; \vec{p}) = \sum_x \langle \mathcal{O}_P(\vec{x}, t) \mathcal{O}_P^\dagger(\vec{0}, 0) \rangle e^{-\vec{p} \cdot \vec{x}} = \sum_m (-1)^{mt} \frac{|\langle 0 | \mathcal{O}_P | P \rangle|^2}{2E_P^{(m)}} e^{-E_P^{(m)} t}. \quad (12)$$

where P represents the meson we want to study and \mathcal{O}_P is the interpolating operator. If we insert a complete set of states, the two-point correlation functions are decomposed into the contributions from different energy levels. The m labels the complete set of states that contribute to the sum. The factor $(-1)^{mt}$ arises only with staggered valence quarks. We are interested in the ground state energy and the excited states contributions can be safely neglected at large enough t . The FNAL/MILC collaborations use different smearing functions for B and K mesons to improve the accuracy of the results [34, 16]. The Cambridge/W&M/Edinburgh group applies the all-to-all propagator technique [35] to improve the signal-to-noise ratio. The ground state mass and energy in the current calculations are well-determined and have sub-percent statistical errors.

Lattice QCD extracts hadronic matrix elements from the three-points correlation function. For example, to determine f_+ and f_0 in $B \rightarrow Kl^+l^-$ decay, we measure $C_{3,\mu}(t, T; \vec{p}_K)$ which is defined as

$$C_{3,\mu}(t, T; \vec{p}_K) = \sum_{\vec{x}, \vec{y}} e^{i\vec{p}_K \cdot \vec{y}} \langle \mathcal{O}_K(0, \vec{0}) V_\mu(t, \vec{y}) \mathcal{O}_B^\dagger(T, \vec{x}) \rangle \quad (13)$$

where $V_\mu = i\vec{s}\gamma_\mu b$. T denotes the location of the sink operator. Similar to the two-point correlation function, if we insert a complete set of states to the three-points correlation function $C_{3,\mu}$, it is decomposed into a sum of energy levels as

$$C_{3,\mu}(t, T; \vec{p}) = \sum_{m,n} (-1)^{mt} (-1)^{n(T-t)} A_\mu^{mn} e^{-E_K^{(m)}t} e^{-M_B^{(n)}(T-t)}, \quad (14)$$

where

$$A_\mu^{mn} = \frac{\langle 0 | \mathcal{O}_K | K^{(m)} \rangle}{2E_K^{(m)}} \langle K^{(m)} | V_\mu | B^{(n)} \rangle \frac{\langle B^{(n)} | \mathcal{O}_B | 0 \rangle}{2M_B^{(n)}}. \quad (15)$$

The A_μ^{00} contains the matrix elements we want. The three-points correlation function also has the contributions from the excited states and opposite-parity states. Different methods are used to extract A_μ^{00} . The FNAL/MILC collaborations apply an iterative averaging trick [34] to suppress the oscillating states contributions. The HPQCD and Cambridge/W&M/Edinburgh group fit the two-point and three-point correlation functions simultaneously to extract A_μ^{00} . They use the constrained fit technique [36, 37], which helps to resolve the information from excited states.

4.2 Chiral-continuum extrapolations and z -expansion fit

Although the strange quark mass (m_s) used in numerical simulations is typically close to its physical value, the light (u, d) quark masses in current simulations are usually larger than their physical values. Very recent new simulations include light quarks with masses at their physical values [39, 40, 41], but they have not yet been used to analyze B meson decays. The FNAL/MILC collaborations perform combined continuum-chiral extrapolations using HMS χ PT. $SU(3)$ HMS χ PT was tested in B and D semileptonic decays [34, 42]. Some preliminary results from FNAL/MILC collaborations suggest the $SU(2)$ HMS χ PT might be a better effective theory in the $B \rightarrow Kl^+l^-$ process. The continuum form factors from lattice data are at the small E_K regime, because the discretization error becomes larger at large E_K in lattice calculations. Moreover, the correlation functions become noisier and HMS χ PT is not reliable at large E_K . To have the form factors on the whole q^2 range, the FNAL/MILC collaborations use the z -expansion to extrapolate the lattice results to low q^2 range.

The z -expansion fit is a model-independent parametrization of form factors on the whole q^2 range. It maps the variable of q^2 to z :

$$z(q^2, t_0) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}, \quad (16)$$

where $t_{\pm} = (M_B \pm M_{K/K^*})^2$. t_0 is selected to keep the absolute value of z smaller than 1. The form factors are then parametrized as:

$$f(q^2) = \frac{1}{B(q^2)\phi(q^2, t_0)} \sum_{k=0}^{\infty} a_k z^k, \quad (17)$$

where $B(q^2) = z(q^2, m_R^2)$ is called the Blaschke factor. m_R denotes the location of the pole in form factors. $\phi(q^2, t_0)$ is called the outer function. In fits of lattice (or experimental) data, the z -expansion is truncated at some finite order. Different choices for B and ϕ yield different expansion coefficients. Generally, ϕ can be chosen to keep the a_i small and hence the truncation error is well controlled. In addition, one can obtain bounds on the a_i based on unitarity and heavy quark power counting. Hence the z -expansion provides us with a systematically improvable description of the q^2 dependence of the form factors. The preliminary results on form factors in $B \rightarrow Kl^+l^-$ from the FNAL/MILC collaborations are summarized in Fig. 1. The continuum form factors at large q^2 ($q^2 \gtrsim 15\text{GeV}^2$) are obtained from chiral-continuum extrapolations with $SU(2)$ HMS χ PT. The form factors at low q^2 ($q^2 \lesssim 15\text{GeV}^2$) are from the z -expansion fit. The systematic errors from chiral-continuum extrapolations, heavy and light quark discretization, renormalization factors, scale determination, light quark mass determination, and finite volume effect are included. The total error at large q^2 is about 5%, which is more accurate than the previous quenched results.

The HPQCD collaboration and Cambridge/W&M/Edinburgh collaborations use modified z -expansion in the extrapolation of lattice form factors to continuum. As described in the last paragraph, the z -expansion fit provides a way to parametrize form factors on the whole q^2 range. The HPQCD collaboration uses a modified version of z -expansion, where heuristic discretization and light quark mass dependent terms are added in order to perform combined chiral, continuum and shape fits [43]. The HPQCD collaboration applied it on the D semileptonic decays [43] and is planning to use this method for the $B \rightarrow Kl^+l^-$ process. The Cambridge/W&M/Edinburgh collaborations also use the same method for the extrapolations in $B \rightarrow K/K^*l^+l^-$ decays. The preliminary results from these two groups are summarized in Fig. 2. The left panel is an example for $f_{+,0,T}$ in the $B \rightarrow Kl^+l^-$ process from the HPQCD collaboration. The HPQCD collaboration measures form factors with 1% accuracy at large q^2 on the $a \approx 0.12\text{fm}$ MILC lattice ensembles (only statistical error is included here.). More measurements will be done in the future on the $a \approx 0.09\text{fm}$ MILC ensembles. Similarly, the right panel is the preliminary result of the $T_{1,2}$ in $B \rightarrow K^*l^+l^-$ process from the Cambridge/W&M/Edinburgh collaborations.

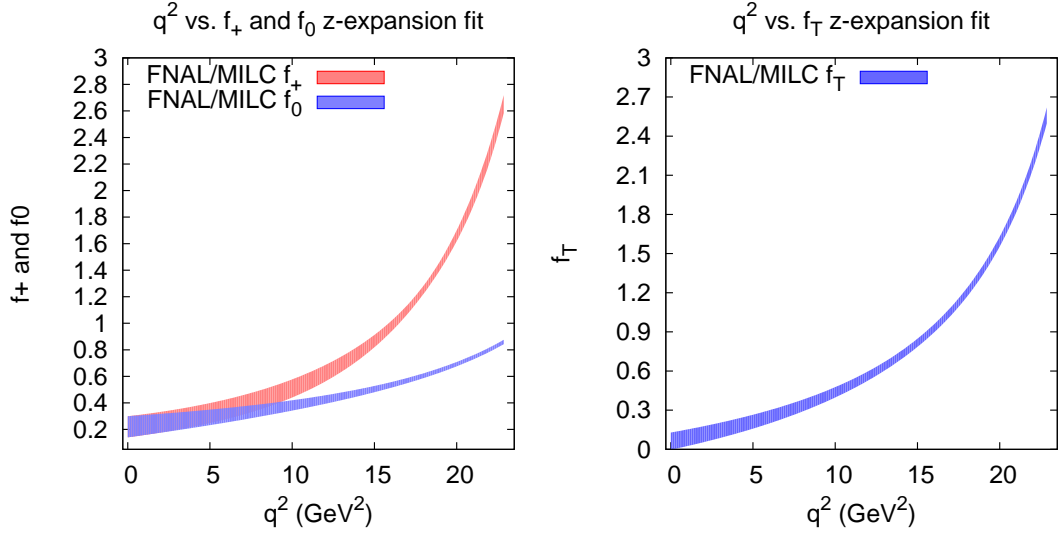


Figure 1: Preliminary result of $f_{+,0}$ (left) and f_T (right) on the whole q^2 range from FNAL/MILC collaborations. They obtain the form factors directly in the range $q^2 \gtrsim 15\text{GeV}^2$, and extrapolate them using the z -expansion into the region of $q^2 \lesssim 15\text{GeV}^2$.

5 Summary

Lattice QCD can calculate the form factors in $B \rightarrow K/K^* l^+ l^-$ decays from first principles. Previous quenched calculations obtained the form factors with an uncertainty of 15%-20% and over a range of q^2 that was limited to large values. The FNAL/MILC collaborations, HPQCD collaboration, and Cambridge/W&M/Edinburgh collaborations are working on the new calculations with dynamical QCD configurations, which include the sea quark effect. The analysis techniques like constrained fit, $SU(2)$ HMS χ PT, and modified z -expansion are used in these new calculations. The preliminary $B \rightarrow K l^+ l^-$ form factors from FNAL/MILC collaborations have a total (combined statistical plus systematic) accuracy of about 5% at large q^2 . Their final results will include a detailed and complete systematic error budget. We can expect increasingly more precise and interesting results from lattice QCD calculation of these rare decays in the future. Sufficiently accurate theoretical predictions of the form factors are essential for making maximal use of experimental measurements, and may yield interesting constraints on new physics in the future.

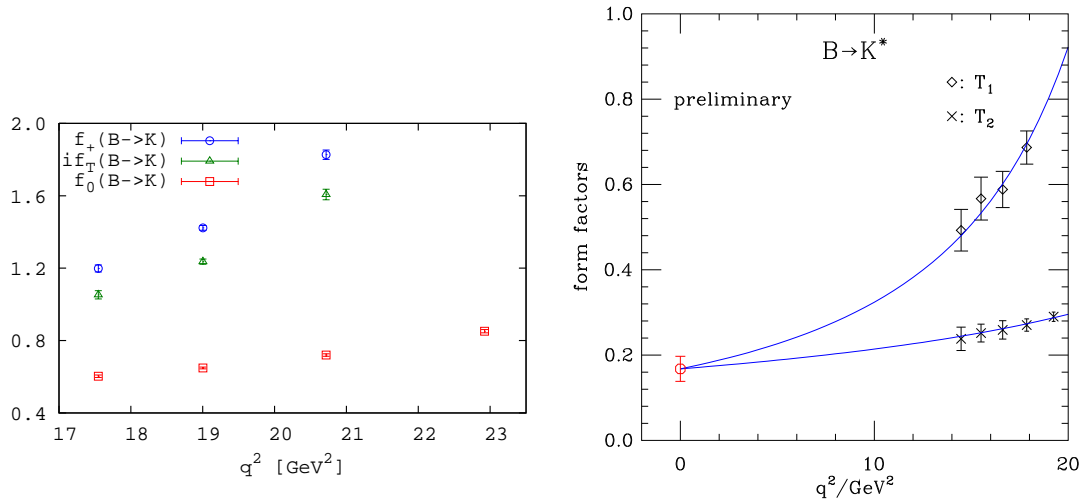


Figure 2: The left panel is from C. Bouchard in the HPQCD collaboration. It is the preliminary result of $B \rightarrow Kl^+l^-$ $f_{+,0,T}$ measured on the lattices. Only statistical error is included and it is about 1% accuracy. The right panel is quoted from Ref. [20]. It is the preliminary result of the $T_{1,2}$ in $B \rightarrow K^*l^+l^-$ process from Cambridge/W&M/Edinburgh collaborations.

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